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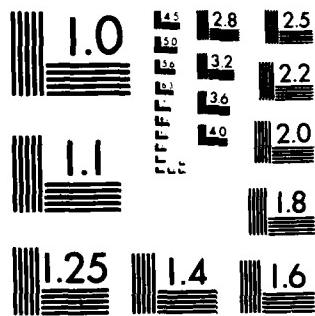
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STRESS ANALYSIS FOR ANISOTROPIC HARDENING IN  
FINITE-DEFORMATION PLASTICITY

E. H. Lee, R. L. Mallett, Rensselaer Polytechnic Institute, Troy, N. Y.  
T. B. Wertheimer, MARC Analysis Research Corp., Palo Alto, Ca.

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Department of Mechanical Engineering  
Rensselaer Polytechnic Institute  
Troy, N.Y. 12181

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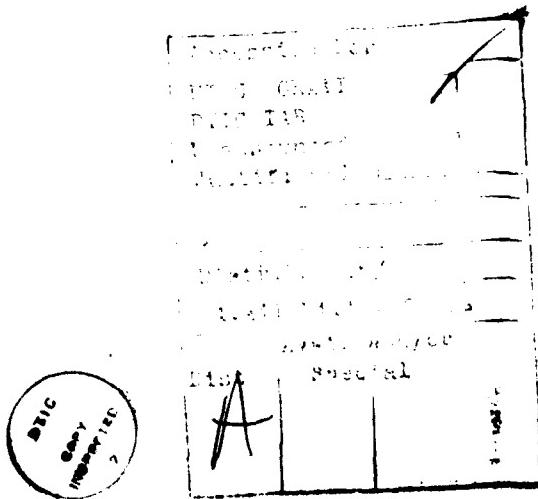
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20. ABSTRACT - continued

A theory is proposed which calls for a modified Jaumann derivative based on the spin of specific material directions associated with the kinematic hardening. This eliminates the spurious oscillation. General anisotropic hardening is shown to require a similar approach.

Though in engineering practice most current stress evaluation for plasticity at finite strains now assume isotropic hardening, the Bauschinger effect can be important and is being more widely incorporated. This paper shows that current finite element computer programs can then generate huge errors. Modification can be made without total restructuring of the program. Further investigation of continuum theory and micromechanics is needed for final resolution.



Stress Analysis for Anisotropic Hardening in  
Finite-Deformation Plasticity

E. H. Lee and R. L. Mallett, Rensselaer Polytechnic Institute

and

T. B. Wertheimer, MARC Analysis Research Corp., Palo Alto, Ca.

Abstract

Kinematic hardening represents the anisotropic component of strain hardening by a shift of the center of the yield surface in stress space. The current approach in stress analysis at finite deformation includes rotational effects by using the Jaumann derivatives of the shift and stress tensors. This procedure generates the unexpected result that oscillatory shear stress is predicted for monotonically increasing simple shear strain.

A theory is proposed which calls for a modified Jaumann derivative based on the spin of specific material directions associated with the kinematic hardening. This eliminates the spurious oscillation. General anisotropic hardening is shown to require a similar approach.

1. Introduction

In a intriguing paper [1], Nagtegaal and de Jong evaluated the stresses generated by simple shear to large deformation in elastic-plastic and rigid-plastic materials which exhibit anisotropic hardening. In conformity with current practice for finite deformation in the case of kinematic hardening, they used an evolution equation for the back stress or shift tensor  $\underline{\alpha}$  (the current center of the yield surface) which relates the Jaumann derivative of  $\underline{\alpha}$  to the plastic strain rate. This incorporates aspects of finite deformation and ensures objectivity of the evolution equation under rigid-body rotations. For a material which

strain hardens monotonically in tension they obtained the unexpected result that the shear traction grows to a maximum value at a shear strain  $\gamma$  of the order unity and then oscillates with a period of about six as the strain increases.

Study of the analytical structure of the kinematic hardening law shows that, in the case of simple shear, the use of the conventional Jaumann derivative causes the shift tensor  $\underline{\alpha}$  to rotate continuously and this generates oscillations in the stress field. However, the back-stress  $\underline{\alpha}$  is a residual stress generated by deformation of the heterogeneous structure of crystallites and hence is embedded in the material. Thus for simple shear the total angular rotation of  $\underline{\alpha}$  must be limited since in simple shear, as pointed out in the following section, no lines of material elements ever rotate by more than  $\pi$  radians. A modified theory is presented which eliminates this anomaly and yields a monotonically increasing shear traction for the problem under discussion.

## 2. The Kinematics of Simple Shear

Using rectangular Cartesian coordinates for the configuration at time  $t$ , a simple shear in the  $x_1$  direction is defined, as depicted in Fig. 1, with displacements

$$u_1 = ktx_2, \quad u_2 = u_3 = 0. \quad (1)$$

The corresponding velocity field is

$$v_1 = kx_2, \quad v_2 = v_3 = 0 \quad (2)$$

having the velocity gradient tensor  $\underline{L}$  with symmetric part  $\underline{D}$ , the rate of deformation, and anti-symmetric part  $\underline{W}$ , the spin.

$$\underline{L} = \frac{\partial v_i}{\partial x_j} = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underline{D} = \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underline{W} = \begin{bmatrix} 0 & k/2 & 0 \\ -k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3)$$

The velocity field is thus steady with constant rate of shear strain  $\dot{\gamma} = k$  and constant spin  $W$  with angular speed  $k/2$ .

Because the velocity field is linear in  $\underline{x}$ , straight material lines remain straight and, for example, the initially square figure illustrated in Fig. 1 is deformed into a sequence of parallelograms. The velocity gradient is uniform over the body so that the angular velocity of any line of particles in the  $(x_1, x_2)$  plane depends only on its current orientation angle  $\theta$  (see Fig. 1) and is given by

$$\dot{\theta} = -k \sin^2 \theta \quad (4)$$

It is evident that the line of particles initially on  $OA_0$  in Fig. 1 approaches the  $x_1$  axis as  $t \rightarrow \infty$ . Moreover, the largest total rotation of any line of particles is less than  $\pi$ , this bound corresponding to the initial inclination  $\theta_0 = \pi - \epsilon$ ,  $0 < \epsilon \rightarrow 0$ .

Note that the angular velocity of the material lines  $\theta = \pi/4$  or  $3\pi/4$ , which coincide instantaneously with the principal directions of the deformation rate tensor  $D$ , is  $k/2$ , equal to the spin as it should be. This is also the average of the angular velocities over all directions in the current configuration.

### 3. The Currently - Adopted Kinematic-Hardening Analysis for Finite Strain

The back stress  $\underline{\alpha}$ , which prescribes the position of the center of the yield surface in stress space, provides the asymmetry in the yield function between continued and reversed loading needed to incorporate such phenomena as the Bauschinger effect.

For combined kinematic-isotropic hardening [2] with the isotropic-hardening stress measure satisfying a Mises type yield condition, the yield function takes the form

$$(s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij}) = (\underline{s} - \underline{\alpha}) : (\underline{s} - \underline{\alpha}) = (\underline{s} - \underline{\alpha}) \cdot (\underline{s} - \underline{\alpha}) = 2\sigma_0^2(\bar{\epsilon}^P)/3 \quad (5)$$

where  $(\cdot):\cdot$  denotes the trace of the matrix product and  $(\cdot)\cdot(\cdot)$  the dot product in nine-dimensional stress space (it is helpful to keep in mind both of these representations). The matrix or vector  $\underline{s}$  is the stress deviator and  $\sigma_0$  is the tensile value of the isotropic part of the yield stress. The latter depends on the history of plastic deformation as expressed through the generalized plastic strain scalar  $\bar{\epsilon}^P$  given by the growth law

$$\dot{\bar{\epsilon}}^P = \sqrt{2 \underline{D}^P : \underline{D}^P / 3} \quad (6)$$

where  $\underline{D}^P$  is the plastic strain rate.

The growth of the anisotropic part of the yield stress in kinematic hardening is given by the evolution equation for the internal variable  $\underline{\alpha}$

$$\dot{\underline{\alpha}} = \dot{\underline{\alpha}} - \underline{W}\underline{\alpha} + \underline{\alpha}\underline{W} = C(\bar{\epsilon}^P)\underline{D}^P \quad (7)$$

where, for finite-deformation applications, the Jaumann derivative  $\dot{\underline{\alpha}}$  is commonly chosen to replace the material derivative  $\dot{\underline{\alpha}}$  used in infinitesimal displacement theory. This ensures that (7) is objective under superposed time-dependent rigid body rotations.

Large shear strains  $\gamma = kt$  of the order 10 are considered so that elastic strains can be neglected and rigid-plastic theory adopted. Thus the plastic strain rate equals the total strain rate defined in (3)

$$\dot{\underline{D}}^P = \dot{\underline{D}}. \quad (8)$$

The normality condition associated with the yield function (5) determines the flow rule

$$\dot{\underline{D}}^P \propto (\underline{s} - \underline{\alpha}). \quad (9)$$

Thus with  $\dot{\underline{D}}^P$  prescribed by (8) and (3),  $\bar{\epsilon}^P(t)$  can be determined by integrating (6), and  $\underline{\alpha}(t)$  by integrating (7) from the initial condition  $\underline{\alpha}(0) = 0$ . Equations (9) and (5) then determine  $(\underline{s} - \underline{\alpha})$ , so that  $\underline{s}(t)$  can be evaluated.

Such an evaluation was presented in [1] and both it and the cor-

responding elastic-plastic solution resulted in oscillatory stresses.

The rigid-plastic case corresponding to purely kinematic hardening ( $\sigma_0$  constant) and linear hardening in tension (constant tangent modulus  $3C/2$ ) can be evaluated analytically, as pointed out to us by Y. F. Dafalias, to give shear stress

$$\tau = s_{12} = \sigma_0/\sqrt{3} + (C \sin \gamma)/2, \quad \gamma = kt \quad (10)$$

and the non-zero normal stress deviator components

$$s_{11} = -s_{22} = C(1-\cos \gamma)/2. \quad (11)$$

For comparison with the results of a modified theory, these stress variations are shown by the oscillatory curves in Figs. 2 and 3 respectively. The oscillations arise since the spin terms in (7) generate a tensor  $\underline{\alpha}$  which rotates with angular velocity  $k/2$  and, because of (5) with constant  $\sigma_0$ , this causes the components of  $\underline{\alpha}$  and hence of  $\underline{s}$  to oscillate with angular frequency  $k$ , and thus with period  $2\pi$  in  $\gamma = kt$ .

#### 4. A Modified Constitutive Relation

Constitutive relations for anisotropic hardening were initially developed for infinitesimal displacement theory, so that, for example, the evolution law (7) for  $\underline{\alpha}$  was expressed in [2] as

$$\dot{\underline{\alpha}} = C(\underline{\epsilon}^P)\underline{D}^P \quad (12)$$

where the superimposed dot denotes the material derivative with the time differentiation performed with respect to axes fixed in space. The term on the right hand side of (12) expresses the influence on the growth of  $\underline{\alpha}$  of the plastic flow currently taking place. However, the effect of the rotation of  $\underline{\alpha}$  due to the deformation of the material in which the back stress is embedded also contributes to the change of  $\underline{\alpha}$ , but this component is neglected in infinitesimal displacement theory in which rotation terms are consistently neglected compared with strain terms.

It was pointed out by Rice [3] that, when the tangent modulus is of the

order of the stress, such an approximation is not justified, even at small strains, and this is often the case in elastic-plastic theory. Thus, for finite deformation, and possibly even for small deformation, the effect of the rotation of the back stress generated by previous plastic flow must be added to the contribution of the plastic flow currently taking place and thus to the right hand side of (12).

It should perhaps be pointed out that more elaborate laws than (12) were developed with the same kinematic restriction of neglecting the change in  $\alpha$  due to its rotation, and these can be written [1]

$$\dot{\alpha}_{ij} = L_{ijkl} D^P_{kl} \quad (13)$$

in terms of the shift operator  $L$  which could depend on  $s$ ,  $\alpha$ ,  $\bar{\epsilon}^P$  and other internal variables determined by the history of deformation. These laws were devised to obtain better agreement with experimental measurements, particularly those involving unloading and reversed loading, but obviously made no contribution towards improving the neglect of the rotation influence. Since both laws (12) and (13) are incremental in form, relating increments or rates of  $\alpha$  and strain, they could be applied at any instant during the deformation history since the required rate variables already occur there.

As mentioned in the Introduction, the back stress  $\alpha$  is embedded in the material as residual stresses generated due to the heterogeneous structure of anisotropic crystallites forming the polycrystalline material. Alternatively this influence can be thought of in terms of dislocations piled up against grain boundaries, or other analogous micro-mechanisms, the mobility of which depends on the strain rate tensor imposed, both with regard to the asymmetry between continued and reversed straining and to the direction of straining in the material. A study of the micro-mechanics of the situation, either at the crystallite level, the

dislocation level, or at both, may be needed to fully understand this question, but such seems not now to be available. However information can be gleaned from the macroscopic theory. In particular, the principal component of  $\underline{\alpha}$  having the largest absolute magnitude produces the major influence on the yield surface and hence on the stress field and is carried in the lines of material elements oriented in the corresponding eigenvector direction. Thus rotation of these lines of material elements may be considered to incorporate the major rotational influence of the back-stress generated by previous plastic flow.

In the case of simple shear, the principal component  $\alpha_{33}$  is zero and since  $\underline{\alpha}$  is a deviator tensor the other two are equal in magnitude and opposite in sign. Thus the choice of the eigenvalue of largest absolute magnitude is not unique and one must therefore look further into the evolution of  $\alpha$ . In the case of kinematic hardening according to (7) or (12),  $\underline{\alpha}$  initially grows parallel to  $D^P$  with the tensile eigenvector at  $\theta = \pi/4$  and the compressive one at  $\theta = 3\pi/4$ . Increments parallel to  $D^P$  are being continually added and for the tensile direction the line of elements which carries the back stress rotates towards the  $x_1$  axis with angular velocity  $k/2$  initially and thereafter with ever decreasing speed as is evident from (4). In contrast, material lines instantaneously coincident with the compressive eigenvector initially rotate with increasing angular velocity as they approach the  $x_2$  axis. The increasingly larger angle which the rotated eigenvector makes with the corresponding tensor increments continuously being added (due to the  $CD^P$  term) inhibits the growth of the compressive eigenvector compared with the tensile one. For example, in simple tension or compression the increments sum in fixed directions and generate the maximum kinematic hardening component (see Hall [6], p. 39 for comparison of tension and compression, with shear).

Such considerations suggest that, in the case of simple shear, the rotation of lines of material elements along the tensile eigenvector of  $\alpha$  play the major role in determining the influence on the evolution equation for  $\alpha$  of the back stress caused by previous plastic flow.

Rotation terms must thus be added to (12) yielding

$$\dot{\alpha} = C(\bar{\varepsilon}^P) D^P + \underline{W}^* \underline{\alpha} - \underline{\alpha} \underline{W}^* \quad (14)$$

where the spin  $\underline{W}^*$  of the line of material elements considered to carry the back stress is given by the angular velocity (4).

Comparing this with (7), the currently accepted evolution equation for kinematic hardening at finite deformation, shows that (7) is equivalent to assuming that the back stress already generated contributes to  $\dot{\alpha}$  according to rotation with constant angular speed  $k/2$  (even though it is embedded in material no directed elements of which ever rotate by more than  $\pi$  radians). The nature of the connection between the elements in an elastic-plastic continuum and the stresses needed to generate such unlimited rotation of the embedded stress clearly rule out the validity of (7) for ductile metals.

The structure of (14) suggests a modified interpretation by writing it in the form

$$\dot{\alpha} = \dot{\alpha} - \underline{W}^* \underline{\alpha} + \underline{\alpha} \underline{W}^* = C(\bar{\varepsilon}^P) D^P \quad (15)$$

where  $\dot{\alpha}$  defines a modified Jaumann derivative associated with the spin  $\underline{W}^*$  of lines of material elements carrying the major influence of the back stress  $\alpha$ . It is shown in the Appendix that  $\dot{\alpha}$  is objective. In fact it is shown that for a spin  $\Omega(t)$  the modified Jaumann derivative

$$\dot{\alpha} = \dot{\alpha} - \Omega \alpha + \alpha \Omega \quad (16)$$

is objective if, under time dependent rigid body rotation expressed by the rotation matrix  $Q(t)$ ,  $\Omega$  transforms as

$$\dot{\Omega} = \dot{Q} Q^{-1} + Q \Omega Q^T \quad (17)$$

This expresses a simple geometrical requirement, namely that the time dependent rotation  $\underline{Q}(t)$  superimposed on the spin  $\underline{\Omega}(t)$  adds the current superimposed spin tensor  $\underline{\underline{Q}}\underline{\underline{Q}}^{-1}$  to the spin  $\underline{\Omega}$  transformed by the rotation which has taken place. This applies in the case where  $\underline{\Omega}$  is the spin of lines of material elements in a deforming body. These matters are discussed in more detail in Section 7.

### 5. Comparison of Solutions

Equation (14) was integrated numerically with the initial condition  $\underline{\alpha}(0) = \underline{0}$  and the result was substituted into (5) and (9) to give the stress variations shown in Figs. 2 and 3 for the shear and normal stresses  $\sigma_{12}$  and  $\sigma_{11} = -\sigma_{22}$ . Purely kinematic hardening was assumed with an initial yield stress  $Y = 207$  MPa (30 ksi) and linear tensile hardening with modulus 310 MPa (45 ksi). These values are appropriate to model an aluminum alloy. The rigid-plastic analysis used implies an incompressible medium and thus stresses are determined only to within an arbitrary hydrostatic pressure since it causes no deformation. This pressure was taken to be zero so that the stresses plotted are stress deviators. Fig. 2 also includes the stress-strain relation in shear for isotropic hardening corresponding to the same tensile behavior. No normal stresses are generated in this case.

It is seen that all the stress-strain curves have a common tangent at zero strain. The two kinematic hardening solutions remain close to each other for strains up to about 0.5 but at larger strains the stresses predicted on the basis of the conventional Jaumann derivative oscillate while the approach suggested in this paper yields a monotonically increasing shear stress-strain curve, with a tangent modulus which decreases as the strain increases.

The stress fields of the conventional approach and the suggested

new one agree for small strains because the eigenvectors of  $D^P$  and  $\alpha$  initially coincide so that  $\underline{W} = \underline{W}^*$  as mentioned in Section 2. With increasing strain, the tensile eigenvector of the new solution approaches an asymptote at  $\theta \sim 15^\circ$ . Thus the tensile strain rate in simple shear becomes inclined at some  $30^\circ$  to the direction in which the maximum tensile yield stress has been generated by induced anisotropy. This angle has been increasing and is consistent with the lessening of the tangent modulus. Such softening has been termed a rotational Bauschinger effect by Jonas [4]. The oscillations predicted by the solution based on the conventional Jaumann derivative are clearly due to the inappropriate use of the spin  $\underline{W}$  to express the influence of the back stress already generated (as discussed in the previous section).

Both [1] and the present paper analyzed purely kinematic hardening without a component of isotropic hardening in order to focus on the effects of anisotropy, although results based on isotropic hardening were presented for comparison. This led to a rather drastic difference in the stress variations given by the two approaches to the kinematic hardening case. A physically more appropriate representation for many materials would be isotropic hardening initially, later accompanied by the growth of a kinematic component. When the conventional Jaumann derivative is used this would introduce an oscillatory component superimposed on a smooth monotonically increasing curve, so that initially a minor ripple with period  $2\pi$  would appear, insufficient to produce a zero tangent modulus. This could thus be observed without an instability developing.

Recently torsion tests have been carried out to a shear strain of 7 on six different ductile metals (copper, brass, nickel, steel and two types of iron) [5]. Tests were carried out for a range of strain rates

and the strain-rate influence was not so marked as to rule out adequate analysis on the basis of rate independent theory. In the low strain-rate isothermal range monotonically increasing stress-strain curves were obtained except for an initial upper yield in the steel and one iron. No indication of a ripple or superimposed oscillation was evident. These results support the concept presented in this paper that the continued rotation of  $\alpha$  predicted by the use of the conventional Jaumann derivative has no physical validity.

#### 6. Elastic-Plastic Stress Analysis

Finite-element elastic-plastic computer programs are available for kinematic hardening and have been considered applicable for stress and deformation analysis at finite deformation since they use the (conventional) Jaumann derivative of stress to account for rotational effects. They were shown in [1] to predict stress oscillations in simple shear. In view of the rigid-plastic solutions discussed in the previous section, use of the modified Jaumann derivative as in the evolution equation (15) can be expected to eliminate the spurious oscillations and provide a satisfactory analysis. Thus the computer codes now in use can be corrected simply by changing the time derivative adopted.

Such a time derivative occurs not only in the evolution equation (15) but also in the plastic flow law where it operates on the stress deviator  $s$  and, for kinematic hardening, takes the form

$$\dot{D}^P = \frac{3}{2h\sigma_0^2} (\underline{s} - \underline{\alpha}) [(\underline{s} - \underline{\alpha}) : \underline{s}]^* \quad (18)$$

where  $h$  is a strain hardening modulus (see [2] for the infinitesimal displacement version). Summation of the elastic and plastic strain rates to give total strain rate yields

$$\dot{D} = \dot{D}^E + \dot{D}^P. \quad (19)$$

Since for elastic-plastic analysis (19) replaces (8),  $D^P$  is not prescribed by the kinematics so that  $(s-\alpha)$  cannot be determined by (9) and (5) but instead must be determined by simultaneous integration of the evolution equation (7) or (15) and the flow law (18).

A modified version of the MARC program was used for such an elastic-plastic analysis and the results fell within 1% of the rigid-plastic solutions shown in Figs. 2 and 3. Since the elastic-plastic model does not represent an incompressible material, the stress (not just the deviator) was evaluated. Because the velocity boundary conditions involve no volume change and the flow law (18) prescribes incompressible plastic deformation, the elastic deformation should also be incompressible and hence the stress deviatoric. With  $\sigma_{33}$  zero,  $\sigma_{11}$  and  $\sigma_{22}$  were found to be opposite in sign and equal in magnitude to within 0.1%. The close agreement of the finite-element solution may seem surprising in view of the severe element distortion at shear strains  $\gamma = 10$ . However, it must be borne in mind that the velocity variation is linear which can be modeled exactly by the finite elements even when distorted.

In an earlier report [7] on this topic a Jaumann type derivative of stress based on the spin of the eigenvector triad of  $\alpha$  was used in the flow law. Since only the part of the spin of  $\alpha$  associated with material rotation needs to be eliminated from the stress-rate loading term, the Jaumann derivative  $\overset{*}{s}$  should have been adopted. Rotation of the anisotropic yield surface about the stress origin can be generated by plastic flow and this component must be associated with non-zero stress rate. This change in the analysis is very significant since only one rate definition now appears in the elastic-plastic theory which greatly simplifies numerical implementation.

## 7. General Theory

Consideration so far has been focused on simple shearing because of the unexpected oscillating shear stress results presented in [1]. However, the concepts involved can be generalized and applied to more complex problems. One can expect problems similar to those encountered in simple shearing to arise often in view of the frequent onset of shear localization or banding associated with plastic deformation which will involve a similar deformation-rotation coupling.

A complete investigation of the micro-mechanics and the structures of possible macroscopic constitutive relations will no doubt be needed to fully understand this phenomenon and to generate a fully tested theory. However, the approach suggested in Section 4 does appear to embody the main essence of the phenomenon and can be generalized to three-dimensional problems.

For simple shear, the deformation (2) occurs in the  $(x_1, x_2)$  shearing plane so that the material elements carrying the back stress  $\alpha$  must rotate about the axis  $x_3$  normal to the plane. Thus only a direction in the plane is needed to determine the associated spin. In three dimensions a component of spin around such a direction may also be needed. Since the main back-stress influence is embedded in the plane defined by the eigenvectors of  $\alpha$  associated with the maximum and minimum eigenvalues, it is suggested that the spin  $\dot{\omega}^*$  should be determined by the angular velocity of the material element line instantaneously coincident with the eigenvector of  $\alpha$  corresponding to the eigenvalue with maximum absolute value, with a spin component around this vector determined by rotation of the plane containing the material elements instantaneously coincident with both eigenvectors.

The general theory of constitutive relations of the type considered

here was developed by Onat and Fardshishah [8] who showed that for objectivity of a relation between  $\underline{\sigma}$ ,  $\underline{D}$  and  $\underline{W}$  involving a tensor state variable  $\underline{\alpha}$  in addition to scalar state variables (which for simplicity will not be specifically indicated in the following representation) it must take the form

$$\dot{\underline{\sigma}} = \underline{g}(\underline{\sigma}, \underline{\alpha}, \underline{D}) + \underline{W}\underline{\sigma} - \underline{\sigma}\underline{W} \quad (20)$$

$$\dot{\underline{\alpha}} = \underline{h}(\underline{\sigma}, \underline{\alpha}, \underline{D}) + \underline{W}\underline{\alpha} - \underline{\alpha}\underline{W} \quad (21)$$

where the functions  $\underline{g}$  and  $\underline{h}$  are isotropic tensor functions. It is common to combine the spin terms with the material-rate terms to obtain

$$\overset{\circ}{\underline{\sigma}} = \dot{\underline{\sigma}} - \underline{W}\underline{\sigma} + \underline{\sigma}\underline{W} = \underline{g}(\underline{\sigma}, \underline{\alpha}, \underline{D}) \quad (22)$$

$$\overset{\circ}{\underline{\alpha}} = \dot{\underline{\alpha}} - \underline{W}\underline{\alpha} + \underline{\alpha}\underline{W} = \underline{h}(\underline{\sigma}, \underline{\alpha}, \underline{D}) \quad (23)$$

The conventional Jaumann derivative thus appears on the left-hand side of each equation. Since large strains are of interest, rigid-plastic theory will be considered in order to simplify the discussion and thus  $\underline{D} = \underline{D}^P$ .

Discussion will be focused on the evolution equation (15) with the understanding that similar considerations apply to the flow law (18). It was pointed out in Section 4 that equation (15) is objective and so it must be expressible in the form (21). This can be independently established by expressing  $\underline{W}^*$  in terms of  $\underline{W}$  and  $\underline{D}$ .

Consider an arbitrary unit vector  $\underline{n}$  and a linear segment of material elements  $\underline{n} ds$ . The relative velocity between the ends of  $\underline{n} ds$  is

$$(\partial v_i / \partial x_j) dx_j = \underline{L} \underline{n} ds \quad (24)$$

The component normal to  $\underline{n}$  determines the spin  $\underline{W}^*$  of that segment to within an arbitrary spin about the segment. Making use of  $\underline{L} = \underline{D} + \underline{W}$  and cancelling  $ds$  throughout gives

$$(\underline{D} + \underline{W})\underline{n} - [\underline{n}^T (\underline{D} + \underline{W})\underline{n}] \underline{n} = \underline{W}^* \underline{n} \quad (25)$$

The  $\underline{W}$  term in the brackets reduces to zero because  $\underline{W}$  is anti-symmetric and introduction of  $\underline{n}^T \underline{n} = 1$  and factoring gives

$$(\underline{W} + \underline{D} \underline{n} \underline{n}^T - \underline{n} \underline{n}^T \underline{D}) \underline{n} = \underline{W}^* \underline{n}$$

so that a solution of (25) is the anti-symmetric matrix

$$\underline{W}^* = \underline{W} + \underline{D} \underline{n} \underline{n}^T - \underline{n} \underline{n}^T \underline{D} \quad (26)$$

and (14) has the form (21).  $\underline{W}^*$  can readily be shown to involve no spin around  $\underline{n}$  introduced by the terms involving  $\underline{D}$ .

For simple shearing,  $\underline{W}^*$  is determined by the spin of a line of material elements instantaneously coincident with an eigenvector of  $\underline{\alpha}$  and (26) gives this rotation about the  $n_{33}$  axis. For general deformation it was suggested above that  $\underline{W}^*$  be defined by the spin of material elements lying along one eigenvector direction, the spin around it being determined by the rotation of the plane determined by material elements along another eigenvector. Since the spins of both material lines are of the form (26) it is clear that the resulting spin will have the form  $\underline{W}$  plus a function of  $\underline{D}$  and hence will lead to a relation of the form (21).

For simple shearing (14) was integrated in the rigid-plastic case and (14) and (18) in the elastic-plastic case using  $\underline{W}^*$  from (4) and this permitted accurate numerical integration because the total rotation of  $\underline{\alpha}$  was less than  $\pi/4$ . Combining all the terms involving  $\underline{D}$  together as in (20) and (21) separates out self-cancelling, oscillating terms which must then combine to yield a monotonic function. This may lead to increasing the inaccuracy in carrying out numerical integration. However in more complicated problems where a simple relation such as (4) does not exist for  $\underline{W}^*$ , it may be important to separate out the  $\underline{W}$  and  $\underline{D}$  variables. Certainly for formulating the structure of the physical theory the resultant rotational effect of the back stress already generated, which is dependent on  $\underline{W}^*$ , is a significant and pregnant concept.

As pointed out in Section 4, generalizations of the simple kinematic hardening law attempt to improve the operator  $L$  in (13) but do not address the effect of finite rotation of the  $\alpha$  generated by previous plastic flow in contributing to  $\dot{\alpha}$ . It would thus be a forlorn hope that the more complicated models would remove the oscillating stress anomaly associated with use of  $W$  in place of  $W^*$  in the evolution equation for  $\dot{\alpha}$  corresponding to (14). However, [1] deduces that the Mróz multisurface model and one involving an additional tensor variable do just that. Study of the laws and methods of evaluation used in [1], reveals that this conclusion arises from shortcomings of the laws selected or the method of evaluation.

The general evolution law used in [1] is

$$\dot{\alpha} = p (m:D^p) \quad (27)$$

where  $m = (s-\alpha)$ , and  $p$  is a tensor which takes on different forms for the three laws studied. In the case of simple shearing, (27) reduces to

$$\dot{\alpha}_{11} = (p_{11}/\sqrt{3} + \alpha_{12})k \quad (28)$$

$$\dot{\alpha}_{22} = (p_{22}/\sqrt{3} - \alpha_{12})k \quad (29)$$

$$\dot{\alpha}_{12} = [p_{12}/\sqrt{3} + (\alpha_{22} - \alpha_{11})/2]k \quad (30)$$

The spin terms which generate the rotation of  $\alpha$  and hence the oscillating stress are the  $\alpha_{12}$  terms in (28) and (29). In simple shear the Bauschinger effect will certainly be most significant for reversed loading in shear, hence  $\alpha_{12}$  will be a dominant component. Thus, in conformity with the physical theory developed in Section 4, one would expect oscillatory stress for all the laws since  $p$  expresses the infinitesimal strain model and is not influenced by the rotation.

Study of the individual cases indicates why the anomaly was limited to the kinematic hardening model. The Mroz multisurface model, for

example, was solved for the limit of an infinite number of closely adjacent surfaces for which the shift rate of each surface in stress space was proportional to  $\underline{\alpha}$  instead of a linear combination of  $\underline{\alpha}$  and  $(\underline{s}-\underline{\alpha})$  away from that limit. Thus the shift rate direction was independent of stress or  $D^P$ , a most unlikely circumstance. Moreover, the  $\underline{\alpha}$  generated as each surface was activated was not accumulated, so that finally the isotropic hardening solution was reproduced - hardly compatible with the anisotropic hardening envisaged. These results therefore do not invalidate the physical concepts on which the theory developed in Section 4 was built nor the error introduced by use of the conventional Jaumann derivative in (15) and (18).

#### 8. Discussion and Conclusions

The modified Jaumann derivative ( $^*$ ), eqn. (15), has some similarity to the Jaumann type derivative ( $^{\wedge}$ ) introduced by Dienes [10] which involves spin associated with the rotation determined by the polar decomposition theorem for the total deformation from the undisturbed configuration. The latter is thus appropriate in formulating the constitutive equation of a material for which the stress depends on the total deformation, for example elasticity when it is expressed in differentiated hypo-elastic form. The claim that ( $^{\wedge}$ ) is also appropriate for plasticity is incorrect however since plasticity obeys an incremental or flow type functional law, closer to a fluid than a solid type, in which the specific configuration of the initial undeformed state does not appear in the incremental or flow type constitutive relation at later times. Deformation type plasticity theory, in which the stress is determined by the total plastic strain, lends itself to simplification through use of the polar decomposition theorem but, except in the case of proportional loading, it is known to be inappropriate

to represent plasticity, particularly at large strains.

Study of the physical situation described in Section 4 shows that the influence of the anisotropy generated by previous plastic flow on the growth of the back stress  $\underline{\alpha}$  arises from a spin associated with directions embedded in the body in which the residual back stress is also embedded. This spin also determines the appropriate Jaumann type derivative of stress in the flow law which must eliminate the contribution to the material derivative of stress which is not associated with the current plastic flowing. The physical model presented considers the influence of the dominant principal component of  $\underline{\alpha}$  but an analogous spin and functional law will arise in the more complete analysis, based on the polycrystalline structure, of the generation and influence of the deformation induced residual back stress.

For the polar decomposition of the deformation gradient  $\underline{F} = \underline{R}\underline{U} = \underline{V}\underline{R}$ , the spin of directions embedded in the body depends not only on  $\dot{\underline{R}}\underline{R}^{-1}$  but also on  $\underline{U}$  or  $\underline{V}$  and their derivatives. For example, in a plane problem of a constant stretch  $\lambda$  in a time dependent direction  $\theta(t)$ ,

$\underline{F} = \underline{U} = \underline{V}$ ,  $\underline{R} = \underline{I}$ , the spin  $\dot{\underline{R}}\underline{R}^{-1}$  is zero, the principal directions of deformation rotate with angular velocity  $\dot{\theta}$  and the lines of material elements coincident with the stretch direction rotate with angular velocity  $\dot{\theta}(1-1/\lambda)$ . Thus,  $(\underline{\alpha})$  based on the spin  $\dot{\underline{R}}\underline{R}^{-1}$  could clearly not express the needed rotational influence of the back stress in this case and hence in general. In the case of principal directions fixed in the body,  $\underline{U}$  can be diagonalized in the form,  $\underline{U} = \underline{P} \Lambda(t) \underline{P}^{-1}$ , where the matrix of eigenvectors  $\underline{P}$  is constant, so that the velocity gradient  $\underline{L}$  becomes

$$\begin{aligned} \underline{L} &= \underline{D} + \underline{W} = \dot{\underline{F}}\underline{F}^{-1} = \dot{\underline{R}}\underline{R}^{-1} + \underline{R}\dot{\underline{U}}\underline{U}^{-1} \underline{R}^{-1} \\ &= \dot{\underline{R}}\underline{R}^{-1} + \underline{R}\underline{P} \dot{\Lambda}^{-1} \underline{P}^{-1} \underline{R}^{-1}. \end{aligned} \quad (31)$$

Since  $\underline{P}$  and  $\underline{R}$  are orthogonal and the diagonal matrix product is commutative the last term in (31) is symmetric and hence  $\dot{\underline{R}}\underline{R}^{-1} = \underline{W}$  which also equals  $\underline{W}^*$ . Only in such a special situation will the modified Jaumann derivative  $(\hat{\cdot})$  be appropriate for finite-deformation plasticity analysis.

Both the Jaumann type derivatives  $(^*)$  and  $(\hat{\cdot})$  as well as the conventional Jaumann derivative fall in the category (A3) discussed in the Appendix. Whereas  $\underline{W}$  in the conventional Jaumann derivative expresses the average angular velocity of all directions around a point and so is an appropriate spin term in the constitutive equation for an isotropic body, for an anisotropic material certain directions will have a special influence and the range of objective derivatives (A3) permits this generality to be incorporated. The particular selection will depend on the physical mechanisms involved as already discussed. In the case of plasticity with isotropic hardening, the stress rate term devolves from the derivative of a stress invariant which is independent of rotation, so that the same contribution will result whichever Jaumann type derivative is selected.

Quite apart from physical appropriateness, it is fortunate that in plasticity analysis it is not necessary to use variables involving the virgin configuration of the material prior to any plastic flow, since many bodies plastically formed in engineering practice have previously been subjected to plastic flow when they were manufactured, for example, forming rolled sheet or extruded rods. The approach presented here for kinematic hardening exhibits the property necessary for application, that measurement of the yield surface (assumed in this case to be consistent with combined isotropic-kinematic hardening) supplies the information needed to formulate the constitutive relation for the analysis of subsequent deformation. The shift tensor  $\underline{\alpha}$  and the isotropic component of the tensile yield stress  $\sigma_0$  comprise all that is needed concerning the previous history of plastic deformation.

We have suggested a generally applicable formulation of kinematic hardening theory and have chosen a simple hypothesis for the macroscopic influence of the micromechanisms which generate the hardening. Clearly a thorough study of this aspect of the theory is called for. This may require an analysis of the micro-mechanics of polycrystalline material involving investigation of the interaction between deforming crystallites, combined with a more general study of the formulation and generalization of macroscopic constitutive relations.

Finite-element computer codes which incorporate kinematic hardening and are considered valid for finite strain are in active current use. In view of the research findings presented here they can involve huge errors. There is thus an urgent need to clarify this question and to generate and demonstrate a reliable means of stress and deformation evaluation in this field of considerable technological importance. To date most forming analyses have been based on isotropic hardening theory, but it is known that the Bauschinger effect, which is exhibited by many structural metals, can have an important influence on such technologically important phenomena as the generation of residual stresses due to forming. This will increase the demand for reliable analysis to incorporate anisotropic hardening into computer codes and hence to complete the research task introduced in this paper.

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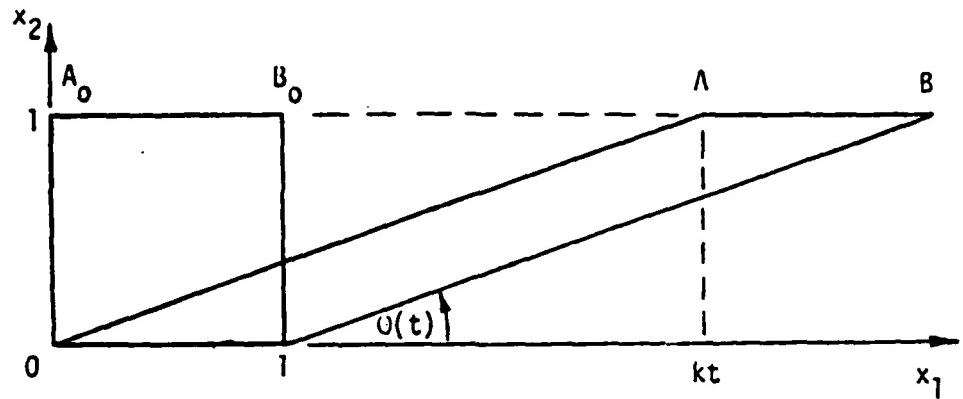


Fig. 1

Simple shear in the  $x_1$  direction

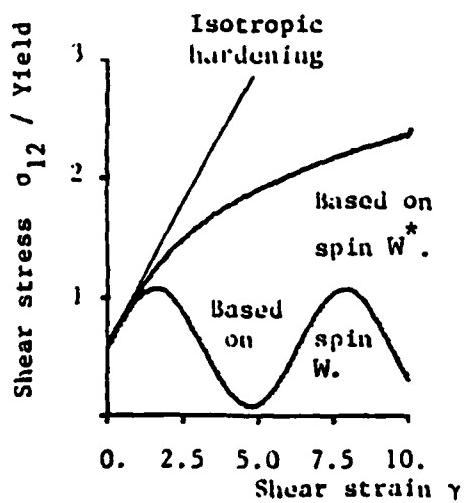


Fig. 2

Shear stress variation

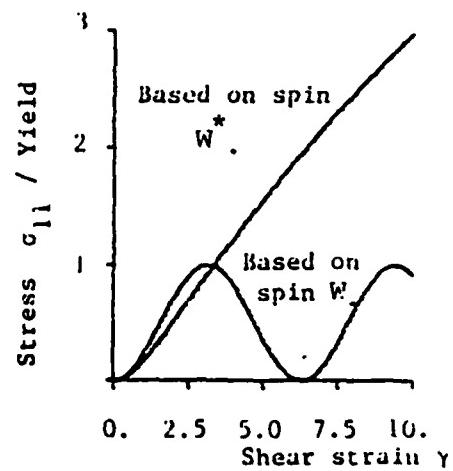


Fig. 3

Normal stress variation

### Appendix : Objectivity

Since the modification of the conventional Jaumann derivative is proposed in this paper, it is perhaps worthwhile to write down explicitly the justification for the objectivity of the analysis. This involves investigating the superposition on the deformed body of a time-dependent rigid-body rotation expressed by the proper orthogonal matrix  $\underline{Q}(t)$  so that the material point coordinates are transformed as  $\underline{x} \rightarrow \underline{Q} \underline{x}$  and (see for example [9])

$$\underline{D} \rightarrow \underline{Q} \underline{D} \underline{Q}^T, \quad \underline{W} \rightarrow \dot{\underline{Q}} \underline{Q}^{-1} + \underline{Q} \underline{W} \underline{Q}^T \quad (\text{A1})$$

The latter transformation expresses the geometrical interpretation of adding the spin  $\dot{\underline{Q}} \underline{Q}^{-1}$  associated with  $\underline{Q}(t)$  to the original spin  $\underline{W}$  transformed by the superposed rotation at that time,  $\underline{Q}(t)$ . Such a transformation clearly applies to the spin of any constituent of the motion not associated with a specific coordinate choice such as a line of material points or the eigenvector triad of  $\underline{\alpha}$  or  $\underline{\Omega}$ .

For a spin  $\underline{\Omega}$  satisfying the transformation

$$\underline{\Omega} \rightarrow \dot{\underline{Q}} \underline{Q}^{-1} + \underline{Q} \underline{\Omega} \underline{Q}^T \quad (\text{A2})$$

the associated Jaumann type derivative of  $\underline{\alpha}$  is

$$\dot{\underline{\alpha}} = \dot{\underline{\alpha}} - \underline{\Omega}\underline{\alpha} + \underline{\alpha}\underline{\Omega} \quad (\text{A3})$$

where  $\underline{\alpha} \rightarrow \underline{Q} \underline{\alpha} \underline{Q}^T$ . The derivative  $\dot{\underline{\alpha}}$  transforms as

$$\begin{aligned} \dot{\underline{\alpha}} &= \dot{\underline{\alpha}} - \underline{\Omega}\underline{\alpha} + \underline{\alpha}\underline{\Omega} \rightarrow \dot{\underline{Q}}\underline{\alpha}\underline{Q}^T + \underline{Q}\dot{\underline{\alpha}}\underline{Q}^T + \underline{Q}\underline{\alpha}\dot{\underline{Q}}^T - (\dot{\underline{Q}}\underline{Q}^T + \underline{Q}\underline{\Omega}\underline{Q}^T) \underline{Q}\underline{\alpha}\underline{Q}^T \\ &\quad + \underline{Q}\underline{\alpha}\dot{\underline{Q}}^T (\dot{\underline{Q}}\underline{Q}^T + \underline{Q}\underline{\Omega}\underline{Q}^T). \end{aligned} \quad (\text{A4})$$

Since  $\dot{\underline{Q}} \underline{Q}^T$  is anti-symmetric, two pairs of terms on the right-hand side of (A4) cancel and the transformed operator becomes

$$\underline{Q} (\dot{\underline{\alpha}} - \underline{\Omega}\underline{\alpha} + \underline{\alpha}\underline{\Omega}) \underline{Q}^T \quad (\text{A5})$$

This result permits a wide choice of Jaumann type derivatives all of which are objective.

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